Characterization of the convolutor and multiplier spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$ by the short-time Fourier transform

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joint work with Norbert Ortner

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Workshop on Functional Analysis Valencia 2015
15–19 June 2015

Short-Time Fourier transform of distributions

Classically the *short-time Fourier transform* is defined as

$$V_g f(x,\xi) = \int_{\mathbb{R}^n} f(y) \mathrm{e}^{-\mathrm{i}\xi y} g(y-x)\,\mathrm{d}y$$

for $f, g \in L^2(\mathbb{R}^n)$. For distributions $f, g \in \mathcal{S}'$ the expression $f(y)g(y-x)$ is defined as the image of

$$f(\xi) \otimes g(\eta) \in \mathcal{S}'(\mathbb{R}^{2n}_{\xi,\eta})$$

under the linear map

$$\mathbb{R}^{2n}_{x,y} \to \mathbb{R}^{2n}_{\xi,\eta}, \xi = y, \eta = x - y.$$

If $f, g \in \mathcal{S}'(\mathbb{R}^n)$ then $V_g f$ is defined by the partial or vector-valued Fourier transform:

$$V_g f = \mathcal{F}_y(f(y)g(y-x)) \in \mathcal{S}'_{x,y}.$$

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A characterisation of $\mathcal{S}(\mathbb{R}^d)$

Theorem (Gröchenig–Zimmermann, 2001)

Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ fixed. Then for $f \in \mathcal{S}'(\mathbb{R}^d)$ the following are equivalent:

1. $f \in \mathcal{S}(\mathbb{R}^d)$.
2. $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$.
3. *For all $n \geq 0$ exists $C_n > 0$ such that*

$$\forall (x,\xi) \in \mathbb{R}^{2d}: \qquad |V_g f(x,\xi)| \leq C_n (1 + |x| + |\xi|)^{-n}.$$

Question

Can we get a similar characterisation of $\mathcal{O}'_C(\mathbb{R}^d)$ and $\mathcal{O}_M(\mathbb{R}^d)$?

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The W_h-transform for distributions

Definition and Proposition

If $h \in \mathcal{S}$ and $F \in \mathcal{S}'_{x,\xi}$ then the W_h-transform

$$W_h \colon \mathcal{S}'_{x,\xi} \to \mathcal{S}'_z, F \mapsto {}_{\mathcal{O}_{C,x}}\langle 1_x, (\mathcal{F}_\xi^{-1} F)(x,z) h(z-x) \rangle_{\mathcal{O}'_{C,x}(\mathcal{S}'_z)}$$

is well-defined, linear and continuous.
The bracket ${}_{\mathcal{O}_{C,x}}\langle \cdot, \cdot \rangle_{\mathcal{O}'_{C,x}(\mathcal{S}'_z)}$ is the $\mathcal{S}'$-valued extension of the evaluation mapping

$$\mathcal{O}_C \times \mathcal{O}'_C \to \mathbb{C}, (\varphi, T) \mapsto T(\varphi)$$

and hence bilinear and hypocontinuous.

Some technical background:

Vector-Valued distributions

Hypocontinuous bilinear mappings

Let E, F and G be three (separated) locally convex spaces and

$$b\colon E \times F \to G$$

a bilinear mapping.
The mapping b is hypocontinuous iff for all bounded set $B \subset E$ and all bounded subsets $B' \subset F$ the mappings

$$B \times F \to G, (e, f) \mapsto b(e, f)$$

and

$$E \times B' \to G, (e, f) \mapsto b(e, f)$$

are continuous

Let E and F be two separated locally convex spaces. We use the following two topologies on the tensor product $E \otimes F$.

$E \otimes_\pi F$ ... finest locally convex topology such that

$$\mathrm{can}\colon E \times F \to E \otimes F, (x, y) \mapsto x \otimes y$$

is continuous.

$E \otimes_\beta F$ ... finest locally convex topology such that

$$\mathrm{can}\colon E \times F \to E \otimes F, (x, y) \mapsto x \otimes y$$

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$E \widehat{\otimes}_\pi F$ and $E \widehat{\otimes}_\beta F$ ... completion of $E \otimes_\pi F$ and $E \otimes_\beta F$, respectively.

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Let E and F be separated locally convex spaces. By $E \otimes_\iota F$ we denote $E \otimes_\iota F$ ... finest locally convex topology such that

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is partially continuous.

We have

$$E \otimes_\iota F \hookrightarrow E \otimes_\beta F \hookrightarrow E \otimes_\pi F$$

and

1. If E and F are barrelled, $E \otimes_\iota F = E \otimes_\beta F$
2. If E and F are Fréchet spaces, $E \otimes_\iota F = E \otimes_\beta F = E \otimes_\pi F$
3. If E and F are (DF) spaces, $E \otimes_\beta F = E \otimes_\pi F$
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Vector-valued distributions and the ε-product

Recall the definition of scalar-valued distributions

$$\mathcal{D}'(\Omega) = \mathcal{L}_b(\mathcal{D}(\Omega), \mathbb{C}) = \{T \colon \mathcal{D}(\Omega) \to \mathbb{C};\ T \text{ linear and continuous}\}$$

Let E be a separated locally convex topological vector space.

Definition

We define the space of E-valued distributions by

$$\mathcal{D}'(\Omega; E) := \mathcal{L}_b(\mathcal{D}(\Omega), E) = \{T \colon \mathcal{D}(\Omega) \to E;\ T \text{ lin. and cont.}\}.$$

For $\Omega = \mathbb{R}^n$, we use the shorter notation

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We want to have not only $\mathcal{D}'(\mathbb{R}^n; E)$ but also other spaces of vector-valued distributions like $\mathcal{S}'(\mathbb{R}^n; E)$.

Observations:

1. $\mathcal{D}'(\mathbb{R}^n; E)$ depends on the (pre-)dual $\mathcal{D}$ of $\mathcal{D}'$ and not directly on $\mathcal{D}'$.
2. We have

$$\mathcal{D}'(E) = \mathcal{L}_b(\mathcal{D}, E) = \mathcal{L}_\varepsilon((\mathcal{D}')'_c, E)$$

This motivates the following definition:

Definition

Let E and F be locally convex spaces. The space

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is called ε-product of E and F.

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A locally convex space $\mathcal{H}$ is called space of distributions if it is contained in $\mathcal{D}'$ with a finer topology.

Definition

Let $\mathcal{H}$ be a space of distributions. We define

$$\mathcal{H}(E) = \mathcal{H} \,\varepsilon\, E.$$

We have $E \otimes F \subset E \,\varepsilon\, F$ and denote by $E \otimes_\varepsilon F$ the space $E \otimes F$ with the topology induced by $E \,\varepsilon\, F$

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Combination with continuous bilinear maps

Theorem (L. Schwartz 1958)

Let $\mathcal{H}$ and $\mathcal{K}$ be normal spaces of distributions and $\mathcal{L}$ be a space of distributions. Moreover let E and F be two separated locally convex spaces. We assume $\mathcal{H}$ to be a nuclear space admitting a nuclear dual space. Let $*\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}$ be a hypocontinuous convolution mapping.
There is a (unique, if $\mathcal{K}$ has the approximation property) bilinear map

$$\overset{*}{\otimes} : \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(E \widehat{\otimes}_\pi F), (S, T) \mapsto \overset{*}{\otimes}(S, T)$$

such that $\overset{*}{\otimes}((S \otimes e), (T \otimes f)) = S * T \otimes e \otimes f$ for all $S \in \mathcal{H}$, $T \in \mathcal{K}$, $e \in E$ and $f \in F$. Moreover the convolution mapping $\overset{*}{\otimes}$ is hypocontinuous with respect to bounded subsets of $\mathcal{H}(E)$ and $\mathcal{K}(F)$.

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The Problem

1. This result only allows the combination of a hypocontinuous mapping with a continuous mapping.
2. In our situation both mappings are not continuous but hypocontinuous.
3. There are (complicated) results for partially continuous bilinear mappings but only for special spaces of vector-valued distributions, e.g. spaces with support restrictions.
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Proposition (B.–Ortner, 2013)

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be complete spaces of distributions (or more general complete locally convex spaces), where $\mathcal{H}$ is nuclear. Let E, F and G be three locally convex spaces, G complete, and

$$u\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L} \text{ and } b\colon E \times F \to G$$

be two hypocontinuous bilinear maps. If one of the assumptions

1. *$\mathcal{H}$ and E are Fréchet spaces*
2. *$\mathcal{H}$ and E are (DF)-spaces*

is satisfied, there is a hypocontinuous bilinear map

$$\genfrac{}{}{0pt}{}{u}{b}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$$

satisfying the consistency property

$$\genfrac{}{}{0pt}{}{u}{b}(S \otimes e, T \otimes f) = u(S,T) \otimes b(e,f).$$

[...]

Proposition (B.–Ortner, 2013)

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be complete spaces of distributions (or more general complete locally convex spaces), where $\mathcal{H}$ is nuclear. Let E, F and G be three locally convex spaces, G complete, and

$$u\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L} \text{ and } b\colon E \times F \to G$$

be two hypocontinuous bilinear maps. If one of the assumptions

1. *$\mathcal{H}$ and E are Fréchet spaces*
2. *$\mathcal{H}$ and E are (DF)-spaces*

is satisfied, there is a hypocontinuous bilinear map

$$\genfrac{}{}{0pt}{}{u}{b}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$$

[...]
If $\mathcal{K}$ satisfies the approximation property $\genfrac{}{}{0pt}{}{u}{b}$ is the unique partially continuous bilinear map satisfying this consistency property.

Proof Sketch

1. Main ingredient: L. Schwartz' "Théorèmes de croisement" yields the existence of a bilinear map

$$\Gamma_{\beta,\beta}\colon (\mathcal{H}\widehat{\otimes}_\beta E) \times (\mathcal{K}\,\varepsilon\, F) \to (\mathcal{H}\widehat{\otimes}_\beta \mathcal{K})\,\varepsilon (E\widehat{\otimes}_\beta F)$$

   which is the unique partially continuous mapping which conincides with the canonial mapping on the tensor products.
2. The assumptions on $\mathcal{H}$ and E yield $\mathcal{H}\widehat{\otimes}_\beta E = \mathcal{H}(E)$.
3. Show that bounded subsets of $\mathcal{H}(E)$ and $\mathcal{K}(F)$ satisfy the conditions of the "Théorèmes de croisement" such that $\Gamma_{\beta,\beta}$ is hypocontinuous.
4. Compose $\Gamma_{\beta,\beta}$ with the ε-product of the continuous linear maps corresponding to u and b.

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Definition and Proposition

If $h \in \mathcal{S}$ and $F \in \mathcal{S}'_{x,\xi}$ then the W_h-transform

$$W_h \colon \mathcal{S}'_{x,\xi} \to \mathcal{S}'_z, F \mapsto {}_{\mathcal{O}_{C,x}}\langle 1_x, (\mathcal{F}_\xi^{-1} F)(x,z) h(z-x) \rangle_{\mathcal{O}'_{C,x}(\mathcal{S}'_z)}$$

is well-defined, linear and continuous.
The bracket ${}_{\mathcal{O}_{C,x}}\langle \cdot, \cdot \rangle_{\mathcal{O}'_{C,x}(\mathcal{S}'_z)}$ is the $\mathcal{S}'$-valued extension of the evaluation mapping

$$\mathcal{O}_C \times \mathcal{O}'_C \to \mathbb{C}, (\varphi, T) \mapsto T(\varphi)$$

and hence bilinear and hypocontinuous.

Proof.

The inclusion $h \in \mathcal{S}$ implies $h(z-x) \in \mathcal{S}_x \widehat{\otimes} \mathcal{O}_{C,z}$ and $F \in \mathcal{S}'_{x,\xi}$ implies $(\mathcal{F}_\xi^{-1} F)(x,z) \in \mathcal{S}'_x \widehat{\otimes} \mathcal{S}'_z$. The previous Proposition yields the unique existence of a hypocontinuous bilinear multiplication

$$(\mathcal{S}'_x \widehat{\otimes} \mathcal{S}'_z) \times (\mathcal{S}_x \widehat{\otimes} \mathcal{O}_{C,z}) \to \mathcal{O}'_{C,x} \widehat{\otimes} \mathcal{S}'_z$$

and hence

$$(\mathcal{F}_\xi^{-1} F)(x,z) h(z-x) \in \mathcal{O}'_{C,x} \widehat{\otimes} \mathcal{S}'_z.$$

The previous Proposition can be applied since the mappings

$$\mathcal{S} \times \mathcal{S}' \to \mathcal{O}'_C, (\varphi, T) \mapsto \varphi \cdot T,$$
$$\mathcal{O}_C \times \mathcal{S}' \to \mathcal{S}', (\varphi, T) \mapsto \varphi \cdot T$$

are hypocontinuous and since $\mathcal{S}'$ is a (DF)-space. □

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Proposition

If $g \in \mathcal{S}'$ *and* $h \in \mathcal{S}$ *then it holds*

$$W_h \circ V_g = \langle g, h \rangle \operatorname{id}$$

on $\mathcal{S}'$*.*

Proposition (B.–Ortner, 2013)

Let $g \in \mathcal{S}$, $g \neq 0$. *Then for* $f \in \mathcal{S}'$ *the following assertions are equivalent:*

1. $f \in \mathcal{O}_C'$,
2. $V_g f \in \mathcal{S}_x \widehat{\otimes} \mathcal{O}_{M,\xi}$ *and*
3. $V_g f$ *is continuous and*

$$\forall k \in \mathbb{N}_0 \, \exists m \in \mathbb{N}_0 \, \exists C > 0 : \quad |(V_g f)(x,\xi)| \leq C(1+|x|^2)^{-k/2}(1+|\xi|^2)^{m/2}$$

for all $(x,\xi) \in \mathbb{R}^{2n}$.

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